Topology and the Language of Mathematics

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Thanks.

Introduction

This book introduces the language of mathematics through point-set topology. Little background in mathematics is assumed.

It is a useful addition to current literature because:

1) The introduction of point-set topology for a primary audience with little to no background in the subject is more effective than some relevant literature in wide use today.

2) The introduction to the language of mathematics is more accessible to the undergraduate / advanced high school math student than some relevant literature in wide use today.

3) It serves as an excellent accompanying text to existing relevant literature.

I hope you find it useful.

- Chris

Part I

Preliminary Material

Chapter 1

Logic

1.1 Remark

X is a car implies X is a car or truck. That is, X is a car $\Longrightarrow$ X is a car or truck. But X is a car or truck does not imply that X is a car. For example, X could be a truck.

1.2 Remark

X is a car and X is blue $\Longrightarrow$ X is blue. But X is blue does not imply that X is a car and X is blue. For example, X could be a blue truck.

1.3 Remark

Suppose that Bob and Mary have had only one child and that child's name is Lucy. Then X is Bob and Mary's child $\Longrightarrow$ X is named Lucy. But X is named Lucy does not imply that X is Bob and Mary's child.

1.4 Remark

X is the state capital of Illinois implies that X is Springfield, IL.
X is Springfield, IL implies that X is the state capital of Illinois.
We can say this more concisely in the following way:
X is the state capital of Illinois if and only if X is Springfield.
X is the state capital of Illinois $\Longleftrightarrow$ X is Springfield.

If we want to show that A is true if and only if B is true, we will often show that A implies B and that B implies A.

1.5 Remark

The negation of the statement 'Cindy is a cat' is 'Cindy is not a cat.' The negation of 'There exists an x in T such that x is blarg.' is 'There does not exist an x in T such that x is blarg.' or equivalently, 'Every x in T is not blarg.'

1.6 Remark

Suppose we want to show that every person in the world has the quality of being blarg. Then it is enough to choose one arbitrary person in the world, and show that person must have the quality of being blarg.

For example: Suppose we want to show that every integer is a rational number. Let x be an integer. Then $x = \frac{x}{1}$.
$\frac{x}{1}$ is rational. Given an arbitrary integer x, we have shown that x must be rational. So every integer is rational.

1.7 Remark

Let P be the following statement: A implies B. The contrapositive of P is the following statement: not B implies not A. A statement is true if and only if its contrapositive is true. Sometimes if you want to show that A implies B, it is easier to show that not B implies not A. For example: The contrapositive of 'John is blarg implies Jason is bloog' is 'Jason is not bloog implies John is not blarg.'

1.8 Remark

Let x be a rational number and let y be an irrational number. Show $x + y$ is an irrational number.

Proof. Since x is rational, $x = \frac{a}{b}$ for some integers a and b, $b \neq 0$. Suppose that $x + y$ is rational. Then $x + y = \frac{c}{d}$ for some integers c and d, $d \neq 0$. So $\frac{a}{b} + y = \frac{c}{d}$. So $y = \frac{c}{d} - \frac{a}{b}$. So $y = \frac{bc-ad}{bd}$. a, b, c, and d are integers. So bc, ad, bd, and $bc - ad$ are integers. So, we have shown that y is rational. But y is not rational. So our assumption that $x + y$ is rational must have been false. So $x + y$ is irrational.

□

This is a proof by contradiction. We assume that something is false and arrive at a contradiction. Then we conclude that what we assumed is false is actually true. We will try not to use proof by contradiction very often, since it is a little awkward.

Chapter 2

Sets

2.1 Definition

A **set** X is a collection of things.

If x is one of the things in X, then x is said to be an **element** of X. This is written $x \in X$. If x is not an element of X we write $x \notin X$. If x and y are elements of X, we write $x \in X$ and $y \in X$ or equivalently $x, y \in X$.

2.2 Example

The set of people whose first name starts with T is a set. We might write this set as: $A = \{x;\ \ x$ is a person with a name that starts with the letter T$\}$ We read this as 'A equals the set of all x such that x is a person with a name and this name starts with the letter T.' (Notice the ; is read 'such that')

My first name is Chris, so I am not an element of this set. So Chris $\notin A$. Suppose you choose a person named Tom. Then he is in this set. We can say Tom $\in A$. Note that A contains millions of elements.

2.3 Example

Let $X = \{0, 1, 2\}$. This set has three elements. Its three elements are the numbers 0, 1, and 2. 0, 1, 2 $\in X$. 3 $\notin X$, car $\notin X$, and 2.012 $\notin X$.

2.4 Definition

Suppose we have two sets, A and B. Suppose every element of A is an element of B. Then we say that A is a **subset** of B (or A is **contained** in B). We write $A \subset B$.

2.5 Notation

For the rest of the book, let $\mathbb{R}$ = the set of all real numbers, $\mathbb{Z}$ = the set of all integers, $\mathbb{N}$ = the set of all positive integers, $\mathbb{Q}$ = the set of all rational numbers.

2.6 Example

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

2.7 Notation

For the rest of this book $\exists$ means 'there exists' and $\forall$ means 'for all'.

2.8 Definition

Let A be the set with no elements. Then A is called the **empty set**. The empty set is often denoted ϕ.

2.9 Example

Let $A = \{x \in \mathbb{R};\ x + 2 > -2 \text{ and } x + 3 < -5\}$

Claim: $A = \phi$

Proof. Assume that $A \neq \phi$. Then $\exists\, t \in A$.
So $t + 2 > -2$ and $t + 3 < -5$.
So $t > -4$ and $t < -8$.
This is bad since there isn't any such $t \in \mathbb{R}$.
So our assumption that $A \neq \phi$ has led to a contradiction.
So $A = \phi$.

□

2.10 Result

Let X be any set.

Claim: $\phi \subset X$

Proof. Suppose that ϕ is not contained in X. That means $\exists\, x \in \phi$ such that $x \in X$. So $\exists\, x \in \phi$. But there is no $x \in \phi$. So our assumption that ϕ is not contained in X is false. So $\phi \subset X$.

□

2.11 Example

Let $X = \{0, 1\}$. Then $\phi \subset X$ (by 2.10). But $\phi \notin X$. X has only two elements, the numbers 0 and 1. ϕ (along with every other set) is not an element of X.

2.12 Example

Let $X = \{\phi, 0\}$. Then $\phi \subset X$ by 2.10. X has two elements, the number 0 and the set ϕ. So $\phi \in X$ and $0 \in X$.

2.13 Remark

This empty set stuff is a little awkward. We will use it because it will be useful to us. It helps in set theory similarly to how having the number 0 and negative numbers in arithmetic is sometimes useful.

2.14 Definition

Let A and B be sets.

Then $A \bigcup B = \{x;\ x \in A \text{ or } x \in B\}$.
This is referred to as the **union** of A and B, or A union B.

Then $A \bigcap B = \{x;\ x \in A \text{ and } x \in B\}$.
This is referred to as the **intersection** of A and B, or A intersect B.

2.15 Example

$\mathbb{Q} \bigcap \mathbb{Z} = \mathbb{Z}$, $\mathbb{Q} \bigcup \mathbb{Z} = \mathbb{Q}$, $\mathbb{Z} \bigcap \mathbb{N} = \mathbb{N}$, $\mathbb{Z} \bigcup \mathbb{N} = \mathbb{Z}$.

2.16 Example

If $A = \{1, 2, 3, 5, 8, 13, 21, \ldots\}$ and $B = \{0, 1, 2, 3, 4, 5\}$ then $A \bigcap B = \{1, 2, 3, 5\}$ and $A \bigcup B = \{0, 1, 2, 3, 4, 5, 8, 13, 21, \ldots\}$

2.17 Remark

In 2.16 we did not list every element of A. A had an infinite number of elements, and we described them in a way we hoped was clear. Another possible way to describe the same set A would have been the following. Let $n_1 = 1$. Let $n_2 = 2$. For each integer $i \geq 3$, let $\mathrm{n}_i = \mathrm{n}_{i-1} + \mathrm{n}_{i-2}$. Then define $A = \{n_k; \quad k \in \mathbb{N}\}$. Whenever you must describe a set, you should choose a way to correctly describe the elements in a set, concisely and clearly. Sometimes it may be logical to list every element of a set. Sometimes it may make more sense to not.

2.18 Definition

Let A and B be sets. $A = B$ means $A \subset B$ and $B \subset A$.

2.19 Example

Let $A = \{\text{cat, dog}\}$, $B = \{\text{parrot, dog}\}$.
$A \bigcap B = \{\text{dog}\}$
$A \bigcup B = \{\text{cat, dog, parrot}\}$

Claim: $\{\text{cat, dog, parrot}\} = \{\text{cat, dog, parrot, cat}\}$

Proof. Let the set on the left hand side of the equals sign be LHS. Let the set on the right hand side of the equals sign be RHS.

LHS $\subset$ RHS:
Every element of LHS is an element of RHS. So LHS $\subset$ RHS.
RHS $\subset$ LHS:
Every element of RHS is an element of LHS. So RHS $\subset$ LHS.

We have shown that LHS $\subset$ RHS and RHS $\subset$ LHS. So LHS = RHS.

$\square$

2.20 Definition

Let X and A be sets. Define $X - A = \{t;\, t \in X \text{ and } t \notin A\}$. $X - A$ is called the **complement** of A in X.

2.21 Example

$\{0, 1\} \subset \mathbb{R}$

$\mathbb{R} - \{0, 1\} = (-\infty, 0) \bigcup (0, 1) \bigcup (1, \infty)$, where
$(-\infty, 0) = \{x \in \mathbb{R};\, x < 0\}$,
$(0, 1) = \{x \in \mathbb{R};\, 0 < x < 1\}$ and
$(1, \infty) = \{x \in \mathbb{R};\, x > 1\}$.

2.22 Example

Let $A = \{x \in \mathbb{R};\, 0 \leq x < 1\}$, $B = \{x \in \mathbb{R}, \frac{1}{2} \leq x \leq 3\}$

We can also write this $A = [0, 1), B = [\frac{1}{2}, 3]$.

Claim: $A \bigcap B = [\frac{1}{2}, 1)$.

Proof. LHS $\subset$ RHS:
Let $x \in A \bigcap B$.
Then $x \in A$ and $X \in B$.
So $0 \leq x < 1$ and $\frac{1}{2} \leq x \leq 3$.
So $x \geq 0, x \geq \frac{1}{2}, x < 1, x \leq 3$.
So $x \geq \frac{1}{2}$ and $x < 1$. So $x \in [\frac{1}{2}, 1)$.

RHS $\subset$ LHS:
Let $x \in [\frac{1}{2}, 1)$.
Then $x \geq 0$ and $x < 1$. So $x \in A$.
And since $x \geq \frac{1}{2}$ and $x \leq 3, x \in B$.
So $x \in A \bigcap B$.
(Why are we done?)

$\square$

2.23 Example

Let $B = \mathbb{R} - \mathbb{Q}$ (the set of irrational numbers).
$\mathbb{Q} \bigcup B = \mathbb{R}$ and $\mathbb{Q} \bigcap B = \phi$

2.24 Problem

Let $X = \mathbb{N}, A = \{2, 4, 6, 8, \ldots\}$
What is $X - A$? Describe this set in three different ways.

2.25 Problem

Let A and B be defined as in 2.22. Show $A \bigcup B = [0, 3]$.

2.26 Definition

If X has exactly n elements for some $n \in \mathbb{N} \bigcup \{0\}$, then we say X is **finite** .

2.27 Example

$\{0, 1, 2, 10\}$ is finite (with 4 elements).
$\mathbb{R}$ and $\mathbb{N}$ are not finite (Why?).
ϕ is finite (with 0 elements).

2.28 Definition

If a set X is not finite, then we say that X is **infinite** .

2.29 Example

$\mathbb{R}$ and $\mathbb{N}$ are infinite.

2.30 Result

Claim: The complement of a union of sets is the intersection of the complements of those sets. (DeMorgan)

Proof. LHS $\subset$ RHS: Let x be in the complement of a union of sets. Then x is not in the union of these sets. So x is not in any of the sets. So, for every set, x is in its complement. So x is in the intersection of the complements of all the sets.

RHS $\subset$ LHS:
Let x be in the intersection of the complements of the sets. Then x is in the complement of each of the sets. So, x is not in any of the sets. So x is not in the union of the sets. So x is in the complement of the union of the sets.

□

2.31 Problem

Show that the complement of an intersection of sets is the union of the complements of those sets. (DeMorgan)

2.32 Result

Let $A \subset X$.

Claim: $X - (X - A) = A$.

Proof. Let $x \in X-(X-A)$. Then $x \in X$ and $x \notin X-A$, so $x \in A$. Let $x \in A$. Then $x \notin X - A$. Since $A \subset X, x \in X$. So $x \in X - (X - A)$. (Why are we done?)

□

2.33 Problem

Let A, B, and C be sets.
Show $(A \bigcap B) \bigcup C = A \bigcap (B \bigcup C) \iff C \subset A$.

Proof. $\implies$
Suppose $(A \bigcap B) \bigcup C = A \bigcap (B \bigcup C)$. We want to show that $C \subset A$. So, let $x \in C$. We want to show that $x \in A$. Since $x \in C$, $x \in (A \bigcap B) \bigcup C$. So $x \in A \bigcap (B \bigcup C)$. So $x \in A$.

$\impliedby$ Suppose $C \subset A$.
We want to show $(A \bigcap B) \bigcup C = A \bigcap (B \bigcup C)$.

LHS $\subset$ RHS:
Let $x \in (A \bigcap B) \bigcup C$. Then $x \in (A \bigcap B)$ or $x \in C$.

Case 1: $x \in A \bigcap B$
Then $x \in A$ and $x \in B$. Since $x \in B, x \in B \bigcup C$. So $x \in A \bigcap (B \bigcup C)$.

Case 2: $x \in C$.
Then $x \in B \bigcup C$. Since $C \subset A, x \in A$. So $x \in A \bigcap (B \bigcup C)$.
So we have shown that when we suppose $C \subset A$,

$(A \bigcap B) \bigcup C \subset A \bigcap (B \bigcup C).$

There is one thing left to show ... notice what that is, and then show it.

□

Chapter 3

Functions

3.1 Definition

Let A and B be sets. A **function** f from A to B is a rule that assigns to every $a \in A$ precisely one $b \in B$. A is called the **domain** of f and B is called the **codomain** of f. When f assigns a to b, we say that $f(a) = b$. We read this 'f of a equals b'.

3.2 Example

Let $f(x) = x + 2 \, \forall \, x \in \mathbb{R}$.

Claim: f is a function from $\mathbb{R}$ to $\mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. $f(x) = x + 2$. $x + 2 \in \mathbb{R}$. And there is only one number in $\mathbb{R}$ that is equal to $x + 2$. So f is a function from $\mathbb{R}$ to $\mathbb{R}$.

□

Showing a function really is a function is often called showing a function is well-defined.

3.3 Notation

Let A and B be sets. If we want to say 'f is a function from A to B', we will often write $f: A \longrightarrow B$. So in 3.2 on the preceding page, $f: \mathbb{R} \longrightarrow \mathbb{R}$.

3.4 Example

A function $f : A \longrightarrow B$ can assign lots of different elements of A to the same element of B. For example, let $f : \mathbb{Z} \longrightarrow \{0, 1, 2, 3\}$, where $f(x) = 1 \, \forall \, x \in \mathbb{Z}$. f really is a function. Every element of $\mathbb{Z}$ gets assigned to exactly one element in $\{0, 1, 2, 3\}$. And lots of integers are being assigned to the same thing. Some functions assign every element of the domain to a different element of the codomain . . .

3.5 Definition

Let A and B be sets, and let $f : A \longrightarrow B$. We say f is **1-1** (read "one to one") or **injective** when $f(x) = f(y)$ implies $x = y$. Or, equivalently, when x $\neq$ y implies that $f(x) \neq f(y)$.

3.6 Example

Let $f : \mathbb{Z} \longrightarrow \mathbb{R}, f(x) = 2x$.

Claim: f is 1-1.

Proof. Suppose $f(x) = f(y)$. $f(x) = 2x$ and $f(y) = 2y$. So $2x = 2y$. So $x = y$. So f is 1-1.

□

3.7 Example

Let $X = \{x \in \mathbb{R};\ x \geq 0\}$. Let $f : \mathbb{R} \longrightarrow X, f(x) = x^2$.

Claim: f is not 1-1.

Proof. $f(1) = 1 = f(-1)$, but $1 \neq -1$. So f is not 1-1. □

3.8 Remark

In our definition of a function we required every element in the domain to be sent to an element of the codomain. However, we didn't require that every element in the codomain get an element of the domain assigned to it. For example, let $f : \{0, 1, 2\} \longrightarrow \{0, 1, 2\}, f(x) = 0\ \forall\, x \in \{0, 1, 2\}$. Then f is a function, but not every element of the codomain gets an element of the domain assigned to it (for example, there is no $t \in \{0, 1, 2\}$ such that $f(t) = 1$). Some functions assign an element of the domain to every element of the codomain.

3.9 Definition

Let A, B be sets and $f : A \longrightarrow B$. When $\forall\, b \in B\, \exists\, a \in A$ such that $f(a) = b$, we say that f is **onto** . Equivalently, we may say f is **surjective** .

3.10 Example

Claim: The function f in 3.2 is onto.

Proof. Let $y \in \mathbb{R}$. Then $y - 2 \in \mathbb{R}$.
And $f(y - 2) = (y - 2) + 2 = y$. So f is onto. (Why are we done?) □

3.11 Example

Claim: The function f in 3.4 is not onto.

Proof. $2 \in \{0, 1, 2, 3\}$. And there is no $x \in \mathbb{Z}$ such that $f(x) = 2$. So f is not onto. □

3.12 Remark

Suppose you have a function $f : A \longrightarrow B$.
If f is 1-1, you might think of A as at least as 'big' as B.
If f is onto, you might think of B as at least as 'big' as A.
See 3.27 on page 35 for an example of why care should be taken in this interpretation.

Some people get excited about functions that are both injective and surjective.

3.13 Definition

Let $f : A \longrightarrow B$. Let f be injective and surjective. Then we say that f is **bijective** . We call f a **bijection** from A to B or from A onto B.

3.14 Example

Let $f : \{0, 1\} \longrightarrow \{0, 1\}$ where $f(0) = 1$ and $f(1) = 0$.

Claim: f is bijective.

Proof. f is both 1-1 and onto. □

3.15 Example

Let $f : \mathbb{Q} \longrightarrow \mathbb{R}, f(x) = x$.

Claim: f is 1-1, but not bijective.

Proof. f is 1-1:
Suppose $f(x) = f(y)$. $f(x) = x$ and $f(y) = y$. So $x = y$.

f is not onto:
$\sqrt{2} \in \mathbb{R}$. But there is no $x \in \mathbb{Q}$ such that $f(x) = \sqrt{2}$. Why? Suppose that there is such an x. Then $f(x) = x = \sqrt{2}$. But we claim that $\sqrt{2} \notin \mathbb{Q}$.

Lemma: $\sqrt{2} \notin \mathbb{Q}$

Proof of Lemma. Suppose that $\sqrt{2} \in \mathbb{Q}$. Then for some $x, y \in \mathbb{Z}$, $\sqrt{2} = \frac{x}{y}$, where $x \neq 0$ and $\frac{x}{y}$ is written in lowest terms (that is, x and y are relatively prime).
So $\sqrt{2}^2 = \frac{x}{y}^2$.
So $2 = \frac{x^2}{y^2}$.
So $2y^2 = x^2$. The square of an odd number is odd, and $2y^2$ is even. So x must not be odd. That is, x is even.
So $x = 2t$ for some $t \in \mathbb{Z}$. So $2y^2 = x^2 = (2t)^2 = 4t^2$.
So we have $2y^2 = 4t^2$.
So $y^2 = 2t^2$.
Since the square of an odd number is odd, and since y^2 is even, y must be not odd. That is, y is even.
So x and y are even. But we assumed that x and y are relatively prime. So this is a contradiction to our assumption that $\sqrt{2} \in \mathbb{Q}$. So $\sqrt{2} \notin \mathbb{Q}$.

□

Our assumption that $\exists x \in \mathbb{Q}$ such that $f(x) = \sqrt{2}$ has led to a contradiction. So there is no such x. So f is not onto. □

3.16 Definition

Let A, B, and C be sets and $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then we can define $h = g \circ f$, $h : A \longrightarrow C$ with $h(x) = g(f(x)) \,\forall\, x \in A$. That is, we take $x \in A$, we use f to send it to $f(x)$ which is an element of B. Then we take this element of B, $f(x)$, and we apply g to it getting $g(f(x))$. We say h is g **composed** with f or the **composition** of g with f.

3.17 Example

Let $f : \mathbb{Z} \longrightarrow \mathbb{Q}$, $g : \mathbb{Q} \longrightarrow X$ where $X = \{x \in \mathbb{R}; x \geq 0\}$.
Let $f(x) = x \,\forall\, x \in \mathbb{Z}$. And let $g(x) = \sqrt{x} \,\forall\, x \in \mathbb{Q}$.
Let $h = g \circ f$.
Then $h(2) = g(f(2)) = g(2) = \sqrt{2}$.
$h(16) = g(f(16)) = g(16) = \sqrt{16} = 4$.

3.18 Definition

Let $f : A \longrightarrow A$. When $f(x) = x \,\forall\, x \in A$, we say that f is the **identity** on A. We will sometimes write $f = id_A$ or $f = id : A \longrightarrow A$.

3.19 Definition

Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$. We say that $f = g$ when $f(x) = g(x) \,\forall\, x \in X$.

3.20 Definition

Let $f : A \longrightarrow B$, $g : B \longrightarrow A$. Let $h = g \circ f$, and $j = f \circ g$. Note that $h : A \longrightarrow A$ and $j : B \longrightarrow B$. When $h = id_A$ and $j = id_B$, we say that f is the **inverse** of g, and g is the inverse of f. That is, $f = g^{-1}$. The use of the phrase 'the inverse of' is ok, because the inverse of a function is unique.

Note that if $f = g^{-1}$, then we automatically have $g = f^{-1}$.

We say that f and g are inverses of each other.

3.21 Result

Let $id : X \longrightarrow X$.

Claim: $id^{-1} = id$

Proof. $id(id(x)) = id(x) = x \; \forall \; x \in X$. So id is its own inverse. □

3.22 Example

Let $A = \{x \in \mathbb{R};\; x \geq 0\}$
Let $f : A \longrightarrow A, f(x) = \sqrt{x} \; \forall \; x \in \mathbb{R}$.
Let $g : A \longrightarrow A, g(x) = x^2 \; \forall \; x \in \mathbb{R}$.

Claim: $f = g^{-1}$

Proof. $f(g(x)) = f(x^2) = \sqrt{(x^2)} = x$
$g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x$ □

3.23 Result

Let $f : X \longrightarrow Y$.

Claim: f^{-1} exists if and only if f is bijective.

Proof. $\Longrightarrow$ Suppose $\exists\, f^{-1} : Y \longrightarrow X$.
f is onto:
Let $y \in Y$. Then $f^{-1}(y) = x$ for some $x \in X$, since f^{-1} is well-defined. And $f(x) = f(f^{-1}(y)) = (f \circ f^{-1})(y) = id_Y(y) = y$.

f is 1-1:
Let $f(x_1) = f(x_2)$
Then $f^{-1}(f(x_1)) = f^{-1}(f((x_2))$
$(f^{-1} \circ f)(x_1) = (f^{-1} \circ f)(x_2)$
But $f^{-1} \circ f = id_X$.
So we have $id(x_1) = id(x_2)$.
So $x_1 = x_2$.
So we now have f is bijective.

$\Longleftarrow$
Let $f : X \longrightarrow Y$ be a bijection.
We want to construct f^{-1}. Let $y \in Y$. Then $\exists\, x \in X$ such that $f(x) = y$, since f is onto. Since f is 1-1, we know this x is unique. So for each y there is exactly one x such that $f(x) = y$. And that's good.

For each $y \in Y$, choose the one $x \in X$ such that $f(x) = y$, and let $f^{-1}(y) = x$. We have defined the function f^{-1}. So we are done. □

3.24 Problem

Let X, Y, and Z be sets.
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be bijective and define $h = g \circ f$. Show h^{-1} exists and $h^{-1} = f^{-1} \circ g^{-1}$.

3.25 Problem

Let $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h = g \circ f$.
Decide whether each of the following four statements is true or false and show why.
1) If h is surjective, then f is surjective.
2) If h is surjective, then g is surjective.
3) If h is injective, then f is injective.
4) If h is injective, then g is injective.

3.26 Problem

Let $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h = g \circ f$.
Show the following:
1) If f and g are injective, then h is injective.
2) If f and g are surjective, then h is surjective.
3) If f and g are bijective, then h is bijective.

3.27 Problem

Construct a bijection from $[0, 1]$ to $(0, 1)$.
Hint: Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ and let $B = \{x; x \in (0, 1) - A\}$. Construct a function f where $f(b) = b \ \forall \ b \ \in \ B$, and $f(\frac{1}{n}) = \frac{1}{n+2} \ \forall \ n \ \in \ \{2, 3, 4, 5, \ldots\}$. You should then define $f(0)$ and $f(1)$ in a useful way, and show that f is indeed a

bijection.

This gives a reason why we should be careful using the word "big" for infinite sets in the manner it is used in 3.12 on page 30. Why?

3.28 Problem

Let $f : A \longrightarrow A$, $g = f \circ f$, and g injective. Show f is injective.

3.29 Problem

Let $id : X \longrightarrow X$. Show id is bijective.

3.30 Definition

Let $A \subset X, f : X \longrightarrow Y$. Define $f(A) = \{y \in Y;\ y = f(x)$ for some $x \in A\}$. We sometimes call $f(A)$ the **image** of A under f.

Let $B \subset Y$. Define $f^{-1}(B) = \{x \in X;\ f(x) \in B\}$. We sometimes call $f^{-1}(B)$ the **inverse image** of B under f.

Note that $f^{-1}(B)$ exists even when no function f^{-1} exists.

3.31 Example

Define $f : \mathbb{R} \longrightarrow \{-1, 1\}$, $f(x) = \begin{cases} -1 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0 \end{cases}$

Note this is read as $f(x)$ is -1 for x less than or equal to 0, and $f(x)$ is 1 for x greater than 0.

$f([-2,1]) = \{-1\}$

$f([-1,0.1)) = \{-1,1\}$

$f^{-1}(\{-1\}) = (-\infty,0]$

$f^{-1}(\{1\}) = (0,\infty)$

Chapter 4

Exam

It is strongly recommended that you do not continue to the next part until the following exam is easy.

1)
Define $x \in \mathbb{Z}$ to be odd if $x = 2t + 1$ for some $t \in \mathbb{Z}$.
Define $y \in \mathbb{Z}$ to be even if $x = 2t$ for some $t \in \mathbb{Z}$.
Show all of the following:
There is no integer that is both even and odd.
The sum of two odd integers is even.
The sum of two even integers is even.
The sum of an odd integer and an even integer is odd.
The square of an odd integer is odd.
The square of an even integer is even.

2) Let $A \subset B$. Show that $A \bigcap B = A$ and $A \bigcup B = B$.

3) Let X be a set. Show that $\phi \bigcap X = \phi$ and $\phi \bigcup X = X$.

4) State the negations of the following two statements:
a) There are no x's in Y such that X is blarg but not bloog.
b) All x's are bloog.

5) Define $X = \{0, \{0\}, 1\}$. Which of these are elements of X
a) $\{1\}$
b) ϕ
c) X

6) Let f be defined as in 3.15 on page 31. We showed f is not bijective. Which of the following exist
a) $f^{-1}(\mathbb{Q})$
b) $f^{-1}(\mathbb{R} - \mathbb{Q})$
c) $f^{-1}(\{2\})$
d) $f^{-1}(\{\sqrt{2}\}$

Part II

Definition of Topology

Chapter 1

Topology

1.1 Definition

Let X be a set. A **topology** on X is a set T whose elements are subsets of X having the following three properties:

1) ϕ and X are open
2) Given any collection of open sets, their union is open
3) Given any two open sets, their intersection is open

Each element of T called an **open** set.

When we have a set X and a topology T on X, we call (X, T) a **topological space** or simply a **space** . We say that X is equipped with the topology T. When it should cause no confusion, we will sometimes just say X is a space (omitting naming the topology).

The definition of a topology may seem a bit awkward. We will do lots of examples to try to make dealing with it easy.

1.2 Notation

Let (X, T) be a space.

The following all mean the same thing:
1) U is an open set in the topological space (X, T)
2) $U \in T$

And, if the topology T is understood:
3) U is open in X
4) $U^{open} \subset X$

And, if both the set X and the topology T on it are understood:
5) U is open

1.3 Example

Let X be a set. Let $T = \{\phi, X\}$

Claim: (X, T) is a topological space.

Proof. 1) $\phi, X \in T$.

2) A collection of open sets either includes X or does not. If it includes X, then the union of this collection is X, which is open. If it does not include X, then every element in the collection is ϕ. And the the union is ϕ, which is open. So for any collection of open sets, we have shown that their union is open.

3) Let $U_1, U_2 \in T$.

Case 1: U_1 or $U_2 = \phi$
Then $U_1 \bigcap U_2 = \phi$, which is open.

Case 2: U_1 and $U_2 \neq \phi$.
Then $U_1 = U_2 = X$. And $U_1 \bigcap U_2 = X$, which is open.

So (X, T) is a topological space. □

1.4 Definition

When a set X is equipped with a topology T and the elements of T are precisely ϕ and X (as in 1.3), T is said to be the **indiscrete topology** .

Note that we have just found a way to turn any set into a topological space. The set X might have numbers, or cars, or matrices as elements. It doesn't matter. We now have a way to turn the set into a space.

1.5 Example

Let X be a set, and let $T = \{U;\ U \subset X\}$.
In other words, let every subset of X be open in X.

Claim: T is a topology on X.

Proof. 1) $\phi \subset X$, so ϕ is open.
$X \subset X$, so X is open.

2) Let $\{X_\alpha\}$ be a collection of open sets of X. Each X_α is contained in X. So the union of these sets is contained in X, and thus open in X.

3) Let $U_1, U_2^{open} \subset X$. $U_1 \bigcap U_2 \subset X$. So $U_1 \bigcap U_2$ is open in X.

$\square$

1.6 Definition

When a set X is equipped with a topology T and every subset of X is an element of T (as in 1.5), then we say that X has the **discrete topology** .

1.7 Example

Let $X = \{0, 1\}$. Let $T = \{\phi, \{0, 1\}, \{0\}, \{1\}\}$.
T is the discrete topology on X. Or equivalently, (X, T) is discrete.

1.8 Definition

Let (X, T) be a space. When $x \in U^{open} \subset X$ (that is, $x \in U$ and $U^{open} \subset X$), then we say U is a **neighborhood** of x.

1.9 Example

Let X $= \{0, 1, 2\}$. Let $T = \{\phi, \{0, 1, 2\}, \{0, 1\}, \{1, 2\}\}$.

Claim: T is not a topology on X.

Proof. $\{0, 1\}$ and $\{1, 2\}$ are open. $\{0, 1\} \bigcap \{1, 2\} = \{1\}$ which is not open. So (X, T) is not a topological space. $\square$

1.10 Problem

Let $X = \{0, 1, 2\}$. Let $T = \{\phi, \{0, 1, 2\}, \{0, 1\}, \{1, 2\}, \{1\}\}$. Show (X, T) is a space.

1.11 Definition

Let (X, T) be a space. Saying x is a **point** of X is equivalent to saying $x \in X$.

1.12 Problem

Let $X = \{$Jason, Luke$\}$ How many different topologies are there on X? Suppose we take a set Y that has two elements, but the elements are not necessarily Jason and Luke. How many different topologies can you place on that set?

1.13 Problem

Let $X = \{$the greek letter π, the number represented by the greek letter π, 3.1$\}$. How many different topologies can be placed on X?

1.14 Problem

In $\mathbb{R}$, for $a < b$, an **open interval** (a, b) is the set of real numbers greater than a and less than b. In other words, $(a, b) = \{x;\ a < x < b\}$. We require $a < b$. If $a \geq b$, then (a, b) is not defined.

Suppose you try to define a topological space $(\mathbb{R}, T)$ where the open sets are ϕ, $\mathbb{R}$ and every open interval (a, b) where

$a, b \in \mathbb{R}$? Is $(\mathbb{R}, T)$ is a topological space? (Hint: the answer is no).

Chapter 2

Standard Topology on $\mathbb{R}$

2.1 Definition

Recall that in 1.14 on the facing page we defined an open interval in $\mathbb{R}$.
Let $T = \{A;\ A = \phi$ or A is a union of open intervals in $\mathbb{R}\}$
Then we call T the **standard topology on** $\mathbb{R}$. We call it a topology because it is one. But we need to show that it is, which we do now ...

2.2 Result

Let T be the standard topology on $\mathbb{R}$.

Claim: $(\mathbb{R}, T)$ is a topological space.

Proof. 1) Unions of open sets are unions of unions of open intervals which are unions of open intervals, thus open.

2) ϕ is open. And for each $n \in \mathbb{N}$, $(-n, n)$ is open. So $\bigcup_{n=1}^{\infty}(-n, n)$ is open (by 1) above). It would be nice if $\bigcup_{n=1}^{\infty}(-n, n) = \mathbb{R}$. Then we will have shown that $\mathbb{R}$ is open.

Lemma: $\bigcup_{n=1}^{\infty}(-n,n) = \mathbb{R}$

Proof of Lemma. LHS $\subset$ RHS:
Let $x \in \bigcup_{n=1}^{\infty}(-n,n)$. Then $x \in (-n,n)$ for some $n \in \mathbb{N}$. So $x \in \mathbb{R}$.
So $x \in$ RHS.

RHS $\subset$ LHS:
Let $x \in$ RHS. Then $x = 0, x > 0$, or $x < 0$.

Case 1: If $x = 0$, then $x \in (-1,1)$ and so $x \in$ LHS.

Case 2: If $x > 0$, then $\exists\, t \in \mathbb{N}$ such that $t > x$. And $x \in (-t,t)$. So $x \in$ LHS.

Case 3: If $x < 0$, then $\exists\, t \in \mathbb{N}$ such that $-t < x$. And $x \in (-t,t)$. So $x \in$ LHS.

So we have shown that RHS $\subset$ LHS.
So we have shown that LHS $=$ RHS.
So we are done with both the lemma and 2). $\square$

3) Let U_1, U_2 be open. Each is a union of open intervals or empty. $U_1 \bigcap U_2$ is a union of open intervals or empty (Why?). So $U_1 \bigcap U_2$ is open.

So $(\mathbb{R}, T)$ is a topological space. $\square$

2.3 Example

Let $\mathbb{R}$ have the standard topology. Let $a \in \mathbb{R}$.
Let $(a,\infty) = \{x \in \mathbb{R};\ x > a\}$.

Claim: (a,∞) is open.

Proof. There are lots of integers bigger than a. Choose one, and call it T. Look at $(a,\ T)$, $(a,\ T+1)$, $(a,\ T+2)$, etc.
Let $A_i = (a, T+i)\ \forall\, i \in N \bigcup \{0\}$.
Each A_i is open. So $\bigcup_{i=0}^{\infty} A_i$ is open.
If we can show that $\bigcup_{i=0}^{\infty} A_i = (a, \infty)$, then we are done.

Lemma: $\bigcup_{i=0}^{\infty} A_i = (a, \infty)$

Proof of Lemma. Let $x \in \bigcup_{i=0}^{\infty} A_i$. Then $x \in A_i$, for some $i \in \{0, 1, 2, \ldots\}$. So $x \in (a, y)$ for some $y \in \mathbb{R}$, $y > a$. So $x \in (a, \infty)$. Let $x \in (a, \infty)$. $\exists\, M, T \in \mathbb{N}$ such that $M > T > x$. $x \in (a, M) = A_{M-T}$. (Why?)
So $x \in \bigcup_{i=0}^{\infty} A_i$.

□

□

2.4 Problem

Let $\mathbb{R}$ have the standard topology. Show $(-\infty, a)$ is open in $\mathbb{R}$.

2.5 Remark

Let $a, b \in \mathbb{R}, a > b$.
$[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$.
$[a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$.
$(a, b] = \{x \in \mathbb{R};\ a < x \leq b\}$.

$[a, b], [a, b), (a, b]$ are all not open in $\mathbb{R}$. It is awkward to show this with the tools we have now. We would have to show that we could not write any of them as a union of open intervals. In Chapter 5 we will introduce some tools that make showing these are not open easy.

2.6 Remark

Note that we have some topologies that we can place on any set (discrete, indiscrete), and one topology (the standard topology on $\mathbb{R}$) that we can only place on $\mathbb{R}$. In what sense does it make sense to think of the discrete and indiscrete topologies on a given set as two extremes?

Next we will introduce one more topology we can place on any set. Then we will introduce a little more topological language and apply it a bunch to all of the topologies that we have discussed.